

# Emergent Geometry from Mode-Structured Evolution

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## **Abstract**

We develop a geometric extension of mode-based analysis in which geometric structures are not postulated a priori, but arise from constraints on admissible operational evolution.

Within this framework, convergence modes encode structured classes of admissible approximation procedures, while mode-compatible evolutions generate corresponding admissibility structures on asymptotic tangent directions.

We show that:

- admissible evolution induces constrained structures of realizable asymptotic directions,
- finite-propagation hyperbolic evolution generates cone-type admissibility structures,
- invariant geometric structures arise from admissibility-preserving transformations,
- Minkowski geometry emerges as the unique isotropic quadratic invariant associated with hyperbolic admissibility structures.

More generally, the framework suggests that geometry should be understood as an invariant regime induced by admissible classes of operational evolution rather than as a primitive background structure.

This provides a unified operational perspective connecting evolution, admissibility, invariance, and emergent geometry.

**Keywords:** mode-based analysis; operational geometry; admissibility structures; mode-dependent evolution; hyperbolic PDE; finite propagation; causal structures; Lorentz invariance; Minkowski geometry; emergent geometry

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# 1 Introduction

Classical geometry typically treats metric and causal structures as primitive objects. In Lorentzian geometry, for example, spacetime structure is introduced through a prescribed quadratic form whose null directions determine causal cones and admissible propagation.

**Relation to previous work.** This paper builds on the framework of mode-based analysis developed in [1], [2].

In Part I, convergence modes were introduced as structured classes of admissible limiting procedures, and mode-dependent differentials were defined as stable limits over such classes. The resulting framework reformulated differentiability and weak stability in terms of operationally admissible convergence structures.

In Part II, these ideas were extended to variational, geometric, and categorical settings through mode tangent cones, mode-dependent metrics, differential forms, and functorial solution clouds. This established a general operational framework in which differential and geometric objects arise from stability with respect to classes of admissible modes.

The present work develops the next step of this program. Rather than assuming metric or causal structures a priori, we investigate how geometric structures emerge from constraints on admissible operational evolution.

In the mode-based framework, operational procedures of approximation and evolution are treated as primary mathematical objects. Convergence modes encode suppression patterns, ordering relations, regularization procedures, admissible perturbations, and asymptotic stability structures.

The central idea of the paper is the following:

- modes themselves do not define geometry,
- admissible classes of evolution induce constraints on realizable asymptotic directions,
- these constraints generate admissibility structures,
- invariant geometric objects emerge only as stable structures preserved under admissibility-preserving transformations.

Within this perspective, causal cones are not fundamental geometric entities. They appear as special admissibility structures associated with isotropic hyperbolic evolution and finite propagation constraints.

More generally, the framework suggests that geometry should be understood not as a primitive background structure, but as an emergent invariant regime generated by admissible operational evolution.

## 2 Modes and Operational Evolution

### 2.1 Mode-Compatible Evolution

**Definition 1.** *Let  $U$  be a space of states and let  $R$  be a class of convergence modes. A mode-compatible evolution is a family of maps*

$$E_R(t) : U \rightarrow U, \quad t \geq 0,$$

such that for every admissible mode  $r \in R$  there exists an approximating evolution  $E_r^{(n)}(t)$  satisfying

$$E_R(t)u = \lim_{n \rightarrow \infty} E_r^{(n)}(t)u$$

whenever the limit exists and is stable under perturbations of  $r$  in the mode topology.

**Remark 1.** This evolution is not assumed independently of the mode structure. It is obtained as a stable limit of admissible operational procedures.

## 2.2 Convergence Modes

A convergence mode is a sequence

$$r = \{(\Delta t_n, \Delta x_n^1, \dots, \Delta x_n^m, \theta_n)\}_{n \in \mathbb{N}},$$

where:

- $\Delta x_n^i$  are admissible increments,
- $\Delta t_n$  are evolution increments,
- $\theta_n$  denotes auxiliary operational parameters.

Modes encode:

- suppression,
- ordering,
- regularization,
- admissible approximation procedures.

## 2.3 Mode Tangent Structure

**Definition 2.** Let  $R$  be a class of admissible modes.

The mode tangent structure at  $p$  is:

$$T_R(p) = \left\{ \lim_{n \rightarrow \infty} \frac{\Delta x_n}{\|\Delta x_n\|} : r \in R \right\}.$$

The normalization norm is used only to identify asymptotic directions and does not represent an emergent geometric metric.

**Remark 2.** The tangent structure depends on admissible operational procedures rather than on a preassigned metric.

Operationally, an element of  $T_R(p)$  represents an asymptotic admissible direction generated by a class of modes.

Unlike classical tangent vectors, these directions are not initially defined through a metric or differentiable manifold structure. Instead, they arise from the asymptotic behavior of admissible operational increments.

Thus the mode tangent structure is induced by operational evolution before any metric geometry is introduced.

## 3 Admissibility Geometry

### 3.1 Admissible Directions

Modes alone do not determine geometric constraints. Such constraints arise only after specifying admissible classes of operational evolution.

**Definition 3.** Let  $\mathcal{E}_R$  denote a class of admissible evolutions compatible with a mode class  $R$ .

The admissibility structure induced by  $\mathcal{E}_R$  is the subset

$$A_R(p) \subseteq T_R(p)$$

consisting of realizable asymptotic directions generated by admissible evolutions.

### 3.2 Finite Propagation Structures

A particularly important case occurs when admissible evolution satisfies a finite propagation constraint.

**Definition 4.** An admissible evolution has finite propagation speed if there exists  $c > 0$  such that:

$$\limsup_{n \rightarrow \infty} \frac{\|\Delta x_n\|}{|\Delta t_n|} \leq c$$

for all admissible modes.

**Remark 3.** This condition is structural rather than metric. It restricts admissible operational evolution without presupposing a spacetime geometry.

### 3.3 Cone-Type Admissibility Structures

Finite propagation induces a distinguished admissibility structure:

$$C_R(p) = \{v = (v^0, \vec{v}) \in T_R(p) : \|\vec{v}\| \leq c|v^0|\}.$$

This structure partitions admissible directions into:

- admissible interior directions,
- limiting directions,
- non-admissible directions.

In the isotropic first-order hyperbolic regime, the resulting admissibility structure takes cone-type form.

**Remark 4.** Cone-type admissibility structures are induced objects and are not fundamental components of the framework.

## 4 Invariant Geometry

### 4.1 Mode-Preserving Transformations

**Definition 5.** A linear transformation

$$L : T_R(p) \rightarrow T_R(p)$$

is admissibility-preserving if:

$$L(A_R(p)) = A_R(p).$$

## 4.2 Invariant Quadratic Structures

We seek quadratic forms compatible with the admissibility structure.

**Definition 6.** *A quadratic form  $Q$  is admissibility-compatible if:*

- $Q(v) = 0$  on the boundary of  $A_R(p)$ ,
- $Q$  is invariant under admissibility-preserving transformations.

## 4.3 Linearization of Admissibility-Preserving Transformations

At the level of the mode tangent structure, admissibility-preserving transformations act on asymptotic directions. Therefore, their first-order action is represented by linear maps

$$L : T_R(p) \rightarrow T_R(p).$$

We restrict attention to transformations preserving:

- the boundary of the admissibility structure,
- the distinction between admissible and non-admissible directions,
- spatial isotropy,
- overall scaling of units.

The allowance of scaling reflects the fact that the normalization of the quadratic form is conventional.

Operational isotropy means that admissible operational procedures do not distinguish preferred spatial directions.

In mode terms, admissibility classes remain invariant under rotations of spatial increments.

## 4.4 Hyperbolic Isotropic Regime

Suppose:

- admissibility has cone-type structure,
- spatial directions are isotropic,
- admissibility-preserving transformations act linearly.

Then the admissibility-preserving transformations generate the standard Lorentz-invariant quadratic structure.

## 4.5 Uniqueness of the Quadratic Invariant

Let

$$C = \{v = (v^0, \vec{v}) : \|\vec{v}\| \leq c|v^0|\}$$

be the admissibility cone induced by finite propagation.

**Lemma 1.** *The boundary*

$$\partial C = \{v : \|\vec{v}\| = c|v^0|\}$$

*is a homogeneous hypersurface.*

*Proof.* If  $v \in \partial C$  and  $\lambda > 0$ , then

$$\|\lambda \vec{v}\| = \lambda \|\vec{v}\| = \lambda c |v^0| = c |\lambda v^0|.$$

Hence  $\lambda v \in \partial C$ . □

**Lemma 2.** *Assuming spatial isotropy, the admissibility-preserving linear group acts transitively on spatial directions of fixed norm.*

*Proof.* Spatial isotropy means that rotations of the spatial components preserve admissibility. The rotation group acts transitively on spheres

$$\{\vec{v} : \|\vec{v}\| = \rho\}.$$

Therefore all spatial directions of fixed norm are equivalent under admissibility-preserving transformations. □

**Theorem 1** (Minkowski Quadratic Form). *Let  $Q$  be a non-degenerate quadratic form whose null set coincides with  $\partial C$  and which is invariant under spatial rotations. Then, up to a non-zero multiplicative constant,*

$$Q(v) = (v^0)^2 - \frac{1}{c^2} \|\vec{v}\|^2.$$

*Proof.* By spatial isotropy,  $Q$  cannot contain preferred spatial directions. Hence it has the form

$$Q(v) = a(v^0)^2 + b\|\vec{v}\|^2$$

for constants  $a, b$ .

The null condition on the boundary of  $C$  requires:

$$Q(v) = 0 \quad \text{whenever} \quad \|\vec{v}\| = c|v^0|.$$

Substituting gives:

$$a(v^0)^2 + bc^2(v^0)^2 = 0,$$

hence

$$a + bc^2 = 0, \quad b = -\frac{a}{c^2}.$$

Therefore

$$Q(v) = a \left( (v^0)^2 - \frac{1}{c^2} \|\vec{v}\|^2 \right).$$

Renaming  $a$  as the overall normalization yields the result. □

**Remark 5.** *Minkowski geometry therefore emerges as a special invariant regime associated with isotropic hyperbolic admissibility structures.*

Modes  $\longrightarrow$  Admissible Evolutions  $\longrightarrow$  Admissibility Structures  $\longrightarrow$  Invariant Transformations  
 $\longrightarrow$  Emergent Geometry

## 5 Beyond Cone Structures

Cone-type admissibility structures represent only one class of operational geometries.

Different classes of admissible evolution induce different admissibility geometries:

- hyperbolic evolution  $\rightarrow$  cone structures,
- diffusive evolution  $\rightarrow$  non-causal admissibility,
- anisotropic suppression  $\rightarrow$  directional admissibility,
- stochastic evolution  $\rightarrow$  probabilistic admissibility regions.

**Remark 6.** *Geometry is therefore secondary to admissible operational evolution.*

### 5.1 Operational Dependence of Geometry

The induced admissibility geometry depends directly on the operational structure encoded by the mode class.

**Frequency suppression.** Suppose admissible modes suppress high-frequency oscillatory increments in selected spatial directions. Then admissible tangent directions become anisotropic, and the resulting admissibility geometry acquires preferred directions.

**Stochastic modes.** Suppose admissible modes contain stochastic perturbations:

$$\Delta x_n \mapsto \Delta x_n + \xi_n,$$

where  $\xi_n$  are random fluctuations.

Then the boundary of admissibility is no longer sharply localized. Instead, one obtains a probabilistically broadened admissibility region whose effective geometry depends on fluctuation statistics.

**Interpretation.** Thus geometry is not fixed independently of operational procedures. Different admissible mode classes generate different effective geometric structures.

**Theorem 2** (Operational Reconstruction Principle). *Let a class of admissible evolutions satisfy:*

- *finite propagation,*
- *operational isotropy,*
- *admissibility invariance,*
- *existence of a non-degenerate quadratic invariant.*

*Then the induced admissibility geometry is Minkowskian up to normalization.*

## 6 Relation to PDE and Numerical Evolution

Hyperbolic PDE naturally generate finite propagation constraints and therefore induce cone-type admissibility structures.

Moreover:

- CFL-type conditions act as admissibility constraints on operational increments,
- numerical schemes induce distinct admissibility geometries,
- stability of numerical evolution corresponds to invariance of admissibility structure under perturbations of modes,
- different approximation procedures may therefore induce different emergent geometries.

Thus admissibility geometry provides a unified language connecting:

- PDE evolution,
- numerical approximation,
- operational observability,
- invariant geometry.

### 6.1 Examples of Admissibility Geometry

Different classes of admissible evolution induce qualitatively different admissibility geometries.

**Hyperbolic evolution.** For hyperbolic PDE with finite propagation speed, admissible increments satisfy:

$$\frac{\|\Delta x_n\|}{|\Delta t_n|} \leq c.$$

This induces cone-type admissibility structures.

**Diffusive evolution.** For diffusive or parabolic evolution, admissible influence propagates without finite-speed constraints. In this case no distinguished cone structure emerges, and admissibility regions become globally spread.

**Anisotropic evolution.** If admissible modes suppress certain directions asymmetrically, the resulting admissibility geometry becomes direction-dependent and anisotropic.

Thus admissibility geometry depends fundamentally on the operational structure of evolution.

## 7 Examples

### 7.1 Hyperbolic Example: The Wave Equation

Consider the one-dimensional wave equation

$$u_{tt} = c^2 u_{xx}.$$

Its characteristic directions satisfy

$$x \pm ct = \text{const.}$$

Thus disturbances propagate with finite speed  $c$ .

In mode terms, admissible increments satisfy

$$\limsup_{n \rightarrow \infty} \frac{|\Delta x_n|}{|\Delta t_n|} \leq c.$$

Therefore the admissibility structure is cone-type:

$$C_R(p) = \{v = (v^0, v^1) : |v^1| \leq c|v^0|\}.$$

This is the simplest example in which operational evolution induces a causal cone.

## 7.2 Parabolic Example: The Heat Equation

Consider the heat equation

$$u_t = \kappa u_{xx}.$$

Unlike the wave equation, the heat equation has infinite propagation speed at the level of the classical continuum model. Consequently, no finite cone-type admissibility structure is induced by the equation itself.

Thus parabolic evolution illustrates that cones are not fundamental in the mode framework. They arise only for specific classes of admissible evolution, such as hyperbolic systems with finite propagation.

## 7.3 Anisotropic hyperbolic evolution.

Consider

$$u_{tt} = a u_{xx} + b u_{yy}, \quad a \neq b.$$

Admissible propagation now satisfies

$$\frac{(\Delta x)^2}{a} + \frac{(\Delta y)^2}{b} \leq (\Delta t)^2.$$

The admissibility boundary becomes anisotropic, and the induced invariant structure is no longer Lorentz-isotropic.

## 8 Limitations and Open Questions

The construction developed here is intentionally limited.

First, the emergence of Minkowski geometry requires additional assumptions: finite propagation, spatial isotropy, linear action on tangent structures, and compatibility with a quadratic invariant. Without these assumptions, other admissibility geometries may arise.

Second, cone structures are not universal. Parabolic, stochastic, anisotropic, or multiscale evolutions may induce non-conical admissibility regions.

Third, the present paper treats only local tangent-level structures. A full global theory would require transition maps, curvature, holonomy, and compatibility between admissibility structures at different points.

Several questions remain open:

- Can admissibility geometries be classified by classes of evolution?
- What replaces Lorentzian geometry for anisotropic or stochastic mode classes?
- How do admissibility structures interact with nonlinear PDE?
- Can curvature be derived from spatial variation of mode classes?

## 9 Toward Global Admissibility Geometry

The present work develops only local admissibility structures arising from mode-compatible evolution.

A global theory would require:

- transition maps between local admissibility structures,
- compatibility conditions between local mode geometries,
- admissibility-preserving parallel transport,
- global curvature and holonomy structures.

In such a framework, global geometry would emerge from the compatibility of local operational admissibility structures rather than from a preassigned manifold geometry.

Gaussian fluctuations may produce probabilistically broadened admissibility boundaries, while heavy-tailed stochastic modes may generate fractal admissibility structures.

Operational curvature may be interpreted as the failure of admissibility-preserving transport to remain path-independent.

## 10 Relation to Existing Approaches

Several existing frameworks relate geometry and causality to operational or dynamical structures.

Causal and order-theoretic approaches appear in Lorentzian causality theory and causal set programs. Hyperbolic PDE naturally induce finite propagation structures through characteristic evolution. Operational viewpoints on geometry also appear in constructive and measurement-based formulations of spacetime.

However, the present framework differs conceptually from these approaches. We do not assume metric, cone, causal order, or spacetime structure a priori. Instead, admissibility geometry is derived from classes of operationally admissible evolution modes.

## 11 Conclusion

We have shown that geometric structures may emerge from admissible operational evolution rather than being postulated as primitive background entities.

Within the framework of mode-based analysis:

- convergence modes encode admissible operational procedures,
- admissible evolution induces corresponding admissibility structures,

- invariant geometric and metric objects arise as stable structures preserved under admissibility-compatible transformations.

In the isotropic hyperbolic regime, finite propagation constraints induce cone-type admissibility structures, and the corresponding invariant quadratic geometry is necessarily Minkowskian up to normalization.

From this perspective, causal cones are not primitive geometric objects. They appear as emergent admissibility structures generated by constrained classes of mode-compatible evolution.

More generally, the framework suggests a structural inversion of the classical viewpoint: geometry is not primary, while admissible operational evolution is secondary. Instead, invariant geometry itself emerges as a stable regime induced by admissible classes of operational evolution.

This suggests a broader research program in which:

- different classes of evolution generate different admissibility geometries,
- geometric invariants arise from admissibility-preserving structures,
- global geometry emerges from compatibility between local operational admissibility structures.

In this sense, mode-based analysis provides a possible foundation for viewing geometry not as a fixed background structure, but as an invariant organization of admissible operational dynamics.

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